Energy Analysis of Load-Induced Flutter Instability in Classical Models of Electric Power Networks

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Abstract—**In this paper, we consider the local structure of energy functions for electric power networks near points (parameter values) of incipient flutter instability. Previous work by several investigators clearly indicate the subtle nature of energy functions and energy-like Lyapunov functions when the system exhibits such an instability mechanism. In fact the question of existence of an energy function under these circumstances has been raised. The issue is important because it is now well known that power systems with loads contain such bifurcation points. It is shown, herein, that a local energy function does exist in a sense consistent with the inverse problem of analytical mechanics. However, sufficiently near points of flutter instability the energy function for both stable and unstable systems is not sign definite. Such an energy function can not be used as a Lyapunov function. Nevertheless, it is possible to obtain "natural" Lyapunov functions by combining the energy function with one or more additional first integrals. The analysis is based on the association of the linearized undamped power system with loads with a quadratic Hamiltonian system. General (universal) perturbations of the normal forms of the degenerate quadratic Hamiltonians at such bifurcation points are derived and lead to the stated conclusions. An example is included.**

I. INTRODUCTION

POWER system operating practices have evolved so that transmission networks now function quite differently from customary usage a decade or so ago. It is not surprising that previously uncommon, even unobserved, network instability mechanisms have been linked to recent power system failures in several countries, including France, Japan, and the United States. Such events, although rare, can be severely destructive. Consequently there has emerged a renewed interest in constructing a comprehensive, fundamental understanding of power system stability issues. In the present environment, electric power systems often operate close to stability limits. Therefore, it is necessary to characterize such limits with precision and to fully understand the associated mechanisms of instability in order to assure adequate safety margins. In this paper, we study instabilities associated with the movement of a pair of complex-conjugate eigenvalues into the right half plane. Although somewhat neglected with respect to classical models of power systems, there is growing evidence that this type of instability is easily induced by load variations.

In power system direct stability analysis, the importance of an energy function is clearly evident. Beginning with the work of Magnusson [1] energy analysis has been a recurrent theme over a span of nearly four decades. Nevertheless, there remain many questions about the application of energy functions to systems with loads. Most of the discussion in the literature centers on the issue of transfer conductances in the formulation of energy-like Lyapunov functions (for example, [2]–[5]). The essential difficulty is the same whether constant admittance load models are employed and load buses eliminated, or constant power load models are employed and load buses retained.

Intrinsically more fundamental than the question of the existence of energy-like Lyapunov functions are questions about the energy function itself. If an energy function exists then it is often convenient to use it as the basis for construction of a Lyapunov function for a stable system. Typically, such a Lyapunov function provides sharp estimates of the domain of attraction of a sable equilibrium. Indeed, the energy function can often attribute a useful physical interpretation to the stability boundary and thereby suggest means of evaluating stability margins, e.g., the PEBS method [3]. Moreover, such a Lyapunov function can be used to study the affects of system parameter variations on the geometric properties of the domain of attraction.[1] Should the system lose stability under parameter variations, the energy function, although no longer a Lyapunov function, may provide useful information about the mechanism of instability.

Although attempts to construct exact global energy functions for power systems with loads have not proved

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[1]The idea of using potential functions for the study of power system bifurcations was first introduced by Andronov and Neimark in 1961. They studied a three-machine system with loads confined to generator buses, thereby rendering the dynamical model conservative and the energy function obvious, see Aronovich and Kartvelishvili [7].

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satisfactory, various approximations have been employed in the construction of Lyapunov functions and they have enjoyed some success in applications. On the other hand, local energy functions are easier to identify and they also can provide useful information about the nature of impending instability. In fact, when operating near stability limits, they may yield satisfactory estimates of the stability boundary and, if not, they can be useful first approximations for refined Lyapunov functions which provide improved estimates. Our major goals are to characterize flutter instability in terms of energy properties and to develop local, energy related, parameterized Lyapunov functions which capture the mechanism of instability.

Kwatny, Bahar, and Pasrija [5] provide a construction for a local energy-like Lyapunov function for power systems with transfer conductances at strongly stable equilibria. The term energy-like derives its legitimacy from the fact that the function is the Jacobi first integral of an associated Lagrangian system.[2] An important question is whether it represents an energy function if the system is unstable even though it can no longer be a Lyapunov function. In the event of a simple divergence instability, a single eigenvalue in the right half plane, the answer is yes. However, the construction breaks down at points of flutter instability, i.e., a pair of complex-conjugate roots in the right half plane.[3] Thus, the function introduced in [5] can provide no information regarding loss of stability via bifurcation to a limit cycle.

In this paper, we analyze the properties of energy functions near bifurcation points associated with flutter instability. One important observation is that it is possible to define, in a natural way, a (local) energy function at such points, but it will not be sign definite. Moreover, it is not simply a case of the energy function losing definiteness at the point of bifurcation as is the case with a divergence instability. In the flutter situation, all neighboring systems, stable as well as unstable, are associated with an indefinite energy function. Although it may appear unusual for a stable system to be identified with an indefinite energy function, several important examples are well known. Obviously, the energy function can no longer serve as a Lyapunov function. Nevertheless, it is possible to identify "natural" Lyapunov functions by combining the energy function with other characteristic functions.[4]

One way of establishing an energy function for a power system is via the inverse problem of analytical mechanics.[5] That is, given a set of governing differential equations, determine a variational principle from which they are attainable. The solution to this problem, if one exists, provides, among other things, an energy function.[6] This is the point of view behind the analysis in [5]. In effect, the energy function so obtained now determines the equations of motion. The association of a set of dynamical equations with an energy function is generally nonunique in a fundamental way, even when such an association appears intuitively obvious. It is not surprising that two different energy functions may produce the same set of equations. Much more interesting is the fact that small perturbations of these different functions can produce dynamical systems having vastly different qualitative behavior. Another view of this nonuniqueness is the following. Consider a dynamical system of equations with which has been identified an energy function. The set of system associated with energy functions which are smooth distortions of the original function may not include all neighbors of the original system.[7]

In Section II we describe the classical power system model used in our analysis and review some basic results regarding energy functions and stability. Section III contains a simple example of a three-machine system which serves to motivate and illustrate the key concepts of the paper. An undamped power system with loads is locally, but nonuniquely, equivalent to a Hamiltonian dynamical system. By this we mean that in an appropriately chosen coordinate system the linearized power system model is Hamiltonian. Moreover, the nonuniqueness is typically nontrivial. In Section IV we will define and characterize equivalence classes of Hamiltonian systems associated with power systems at points of potential flutter instability. A given power system model may have a representation in more than one class. Thus it inherits the qualitative characteristics of each class. By associating a Hamiltonian system with a given power system we have identified it with an energy function. A study of the behavior of these systems under perturbations constitutes the remainder of Section IV. It will be seen that only those systems associated with an indefinite energy function can lose stability under perturbations. In Section V we discuss the construction of Lyapunov functions from the quadratic first integrals associated with the linear Hamiltonian system.

II. Mode and Problem Definition

2.1. The Classical Power System Model

Consider a fairly general power system model composed of $n+m+l$ buses where buses $i=1,\cdots,n$ are the internal buses of n generators, buses $i=n+1,\cdots,n+m$ are m (voltage-controlled) PV load buses and buses $i=n+m+1,\cdots,n+m+l$ are l (constant power) PQ load buses. The

[2] For further details on this viewpoint, see Bahar and Kwatny [8], [9].

[3] Narasimhamurthi [4] shows quite clearly that the conventional notion of an energy function is ambiguous at points of incipient flutter instability.

[4] We mean first integrals of the defining differential equations to which can be attributed a physical meaning by their association with system symmetries in accordance with Noether's theorem. Classical versions of this theorem can be found discussed in many texts including Arnold [10]. Extension to nonconservative and constrained dynamical systems have been given by Vujanovic [11] and Bahar and Kwatny [12].

[5] The inverse problem has a long history. A comprehensive reference is the text by Santilli [13].

[6] Solution to the inverse problem produces a Lagrangian, a Hamiltonian, and a Jacobi first integral. Either of the latter may be considered an energy function. The Jacobi first integral is often considered the coenergy function.

[7] This point of view is dramatically exemplified by Narasimhamurthi [4].

interconnecting network is considered to be the equivalent reduced network resulting from the elimination of the constant admittance loads and network internal buses. The dynamical equations of motion may be written[8]

$$M\ddot{\delta} + D\dot{\delta} + f_1(\delta, \phi, V, \mu) = 0 \tag{2.1a}$$

$$f_2(\delta, \phi, V, \mu) = 0 \tag{2.1b}$$

where M denotes the diagonal matrix of generator rotor inertias, D the damping matrix, δ the n-vector of generator internal bus angles, ϕ the $m+l$-vector of load bus angles, V the l-vector of PQ load bus voltage magnitudes, and μ a k-vector of network and load parameters. The functions f_1: $R^{n+m+2l+k} \to R^n$ and f_2: $R^{n+m+2l+k} \to R^{m+2l}$ are the usual load flow relations.

In the event that the network does not contain PV or PQ loads the variables ϕ, V are absent as is (2.1b). Also if $D = 0$, (2.1) reduces to

$$M\ddot{\delta} + f_1(\delta, \mu) = 0. \tag{2.2}$$

Furthermore, if the reduced network does not include transfer conductances, then the function $f_1(\delta, \mu)$ is derivable from a potential function $U(\delta, \mu)$, i.e.,

$$f_1(\delta, \mu) = -\frac{\partial U}{\partial \delta}. \tag{2.3}$$

It is common to refer to $U(\delta, \mu)$ as the potential energy function and to define the energy function as

$$E(\dot{\delta}, \delta, \mu) = \frac{1}{2}\dot{\delta}^t M\dot{\delta} + U(\delta, \mu). \tag{2.4a}$$

It is also possible to define the Lagrangian

$$L(\dot{\delta}, \delta, \mu) = \frac{1}{2}\dot{\delta}^t M\dot{\delta} - U(\delta, \mu). \tag{2.4b}$$

Note that (2.2) may be derived via Lagrange's equations using the Lagrangian (2.4b) (with an appropriate dissipation function if $D \neq 0$). The energy function (2.4a) is the Jacobi first integral associated with the Lagrange system of (2.4b).

The arguments leading to the energy function (2.4a) for the system (2.2) may be extended to the more general case of the system (2.1) as described by Tsolas *et al.* [25]. In [25] load buses are included but transfer conductances are not. Indeed, the retention of load buses is intended to circumvent the introduction of transfer conductances, an approach suggested by Bergen and Hill [6]. In the present work, we will be concerned with the local characterization of energy functions. Thus our study is based on a linearization of (2.1). The matrix parameters of the linearized representation involve the same complexity (notably asymmetry) regardless of the load model employed. Therefore, our analysis is constructed to admit any mix of constant impedance, voltage controlled and constant power loads.

For simplicity of exposition, we assume that the system either has an infinite bus or that one degree of freedom has been removed by reference to a swing bus or use of center of angle coordinates. Otherwise $f(\delta)$ has a translational symmetry which imposes various technical qualifications which unnecessarily complicates the following discussion. We also assume $D = 0$ unless otherwise stated.

2.2. Stability of Equilibria

Let $(\delta^*, \phi^*, V^*, \mu^*)$ be an equilibrium point of (2.1). Suppose that the equilibrium point is *strictly causal*[9] in the sense that there exist unique functions $\phi(\delta, \mu), V(\delta, \mu)$ satisfying $f_2(\delta, \phi(\delta, \mu), V(\delta, \mu), \mu) = 0$ in a neighborhood of $(\delta^*, \phi^*, V^*, \mu^*)$ with $\phi(\delta^*, \mu^*) = \phi^*$ and $V(\delta^*, \mu^*) = V^*$.

Under these circumstances the linearized dynamics of (2.1) reduce to the form

$$M\ddot{x} + D\dot{x} + Kx = 0 \tag{2.5}$$

where

$$K = \left(D_\delta f_1 - \begin{bmatrix} D_\phi f_1 \\ D_E f_1 \end{bmatrix} \begin{bmatrix} D_\phi f_2 \\ D_E f_2 \end{bmatrix}^{-1} D_\phi f_2 \right)^* \tag{2.6}$$

where $x = \delta - \delta^*$. In general, $M^t = M > 0$ and $D^t = D \geq 0$. The matrix K, however, is not typically symmetric. When K is not symmetric it is not possible to define, even locally, a potential function via (2.3) as a means for composing the energy function (2.4a).

K depends on the parameter μ, both explicitly through the Jacobian matrices in (2.6) and implicitly through the variation of the equilibrium point with μ. In the absence of damping,[10] the equilibrium is stable if the eigenvalues of K are positive and K has a complete set of eigenvectors. The equilibrium is unstable if one or more eigenvalues of K are negative or complex. We may monitor the stability of the equilibrium as μ varies by tracking the eigenvalues of K. Stability is lost when any combination of the following events occur: one or more initially positive eigenvalues of K move onto the negative real axis, one or more pairs of initially positive eigenvalues meet and move off the real axis.

We employ the following terminology.

Definition 2.1: A parameter value μ^* corresponds to a point of *incipient instability* if every sufficiently small neighborhood of μ^* contains values of $\mu \neq \mu^*$ corresponding to both stable and unstable systems (2.5). The value μ^* corresponds to a point of *incipient divergence instability* if the unstable systems have precisely one eigenvalue in the right half plane and to a point of *incipient flutter instability* if the unstable systems have precisely two eigenvalues in the right half plane with nontrivial imaginary part.

Note that the terms divergence and flutter instability are commonly used in the literature, with some ambiguity, but with a meaning essentially consistent with our definition. Points of incipient instability are bifurcation points. A

[8]The details of the model described herein may be found in Kwatny *et al.* [14]. See also Pai [15] and Anderson and Fouad [16].

[9]Further discussion of the notion of causal equilibria may be found in Kwatny *et al.* [14]. See also DeMarco and Bergen [17].

[10]When K is not symmetric the effect of damping is quite subtle. Some results are given by Kwatny *et al.* [5].

value μ^* corresponds to a point of incipient instability only if one or more eigenvalues of K are at the origin of the complex plane or there are nondistinct eigenvalues of K on the positive real axis. A complete analysis of the mechanism of loss of stability and classification of the bifurcation requires consideration of the nonlinear contributions of the function f.

An equilibrium point is *regular* if it is strictly causal and if there exist unique functions $\delta_e(\mu)$, $\phi_e(\mu)$, and $V_e(\mu)$ satisfying $f(\delta_e(\mu), \phi_e(\mu), V_e(\mu), \mu) = (f_1^t, f_2^t)^t = 0$, in some neighborhood of $(\delta^*, \phi^*, V^*, \mu^*)$, with $\delta_e(\mu^*) = \delta^*$, $\phi_e(\mu^*) = \phi^*$ and $V_e(\mu^*) = V^*$. An equilibrium point is not regular only if K has one or more eigenvalues at the origin. This case is considered by Kwatny *et al.* [14]. We are interested herein in the situation whereby stability is lost because a single pair of positive real eigenvalues of K meet and move off of the real line. This corresponds to a conjugate pair of system eigenvalues moving from the imaginary axis into the right half plane, i.e., a flutter instability. Such a situation does occur in power system models with loads as evidenced by Narasimhamurthi [4], Abed and Varaiya [18], and Alexander [19]. We provide another example below.

Our objective is to characterize system energy functions at bufurcation points of incipient flutter instability. We will associate an energy function with (2.5) in a manner consistent with the discussion following (2.2). Although this appears to be a plausible means of defining an energy function it by no means provides a unique result. As we shall demonstrate, there are very often several Lagrangians, (nontrivially) different from each other, which produce the same equations of motion.[11] It seems equally reasonable to identify any of the corresponding Jacobi first integrals (or, equivalently, the corresponding Hamiltonians) as the energy function.

The consequence of using any one of the alternative energy functions will be seen to have implications that are quite profound. The choice of an energy function carries with it presumptions about the underlying structure of the system that are not apparent in, and distinct from, the equations of motion. In fact, it will be shown that different energy functions imply different system behaviors with respect to perturbations.

III. An Example

We consider a simple example in order to motivate and illustrate the concepts developed herein. Consider the system illustrated in Fig. 1. The admittance matrix is

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} -j(a+b) & ja & jb & 0 \\ ja & -j(a+c) & 0 & jc \\ jb & 0 & -j(b+d) & jd \\ 0 & jc & jd & 3+jg \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

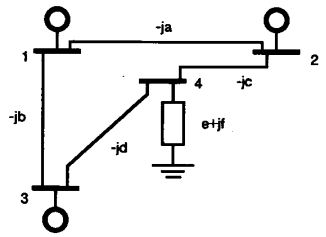

Fig. 1.

where $g = f - c - d$. Since $I_4 = 0$, we obtain $V_4 = (-1/(e - jg))(jcV_2 + jdV_3)$ and the reduced bus relations are

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} C_{11} + jD_{11} & C_{12} + jD_{12} & C_{13} + jD_{13} \\ C_{12} + jD_{12} & C_{22} + jD_{22} & C_{23} + jD_{23} \\ C_{13} + jD_{13} & C_{23} + jD_{23} & C_{33} + jD_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where $C_{11} = C_{12} = C_{13} = 0$, $D_{11} = -(a+b)$, $D_{12} = a$, $D_{13} = b$, $C_{22} = c^2e/(e^2+g^2)$, $D_{22} = -[(a+c)(e^2+g^2) + c^2g]/(e^2+g^2)$, $C_{23} = cde/(e^2+g^2)$, $D_{23} = -cdg/(e^2+g^2)$, $C_{33} = d^2e/(e^2+g^2)$, $D_{33} = -[(b+d)(e^2+g^2) + d^2g]/(e^2+g^2)$. The swing equations can be written:

$$\ddot{\delta}_1 + D_{12}\sin(\delta_1 - \delta_2) + D_{13}\sin(\delta_1 - \delta_3) + C_{12}\cos(\delta_1 - \delta_2)C_{13}\cos(\delta_1 - \delta_3) = P_1 \quad (3.1a)$$

$$\ddot{\delta}_2 + D_{12}\sin(\delta_2 - \delta_1) + D_{23}\sin(\delta_2 - \delta_3) + C_{12}\cos(\delta_2 - \delta_1) + C_{23}\cos(\delta_2 - \delta_3) = P_2 \quad (3.1b)$$

$$\ddot{\delta}_3 + D_{23}\sin(\delta_3 - \delta_2) + D_{13}\sin(\delta_3 - \delta_1) + C_{23}\cos(\delta_3 - \delta_2) + C_{13}\cos(\delta_3 - \delta_1) = P_3 \quad (3.1c)$$

where we have taken $J_i = 1$ and $V_i = 1$, for $i = 1,2,3$. If we take bus 1 as a swing bus and define the variables $\theta_1 = \delta_2 - \delta_1$, $\theta_2 = \delta_3 - \delta_1$, then the swing equations can be reduced to

$$\ddot{\theta}_1 + 2D_{12}\sin(\theta_1) + D_{23}\sin(\theta_1 - \theta_2) + D_{13}\sin(\theta_2) + C_{23}\cos(\theta_1 - \theta_2) - C_{13}\cos(\theta_2) = \Delta P_1 \quad (3.2a)$$

$$\ddot{\theta} + 2D_{13}\sin(\theta_2) + D_{23}\sin(\theta_2 - \theta_1) + D_{12}\sin(\theta_1) + C_{23}\cos(\theta_2 - \theta_1) - C_{12}\cos(\theta_1)\} = \Delta P_2 \quad (3.2b)$$

where $\Delta P_1 = P_2 - P_1$ and $\Delta P_2 = P_3 - P_1$.

The Jacobian, $D_\theta f$, is

$$D_\theta f = \begin{bmatrix} \alpha + z & x + y \\ x - y & \alpha - z \end{bmatrix} \quad (3.3)$$

[11] Note that even $U(\delta)$ as defined by (2.3) is not unique. However, this is not the essential issue as will be evident below.

TABLE I

Case	C_{23}	D_{12}	D_{13}	D_{23}	ΔP_1	ΔP_2	P_1	θ_1	θ_2	λ
1	0	.5	.5	.5	0	0	0	0	0	1.5±0.0
2	2	1	.5774	.5774	4.016	2.913	-1.180	1.092	.5313	1.448±0.020
3	2	1	.5774	.5774	4.034	2.894	-1.161	1.059	.5254	1.486±0.010
4	2	1	.5774	.5774	4.042	2.887	-1.155	1.047	.5236	1.500±0.000
5	2	1	.5774	.5774	4.072	2.857	-1.130	1.003	.5188	1.550±j0.181

$C_{12} = 0, C_{13} = 0$

where

$$\alpha = D_{12}\cos(\theta_1) + D_{13}\cos(\theta_2) + D_{23}\cos(\theta_1 - \theta_2) \tag{3.4a}$$

$$z = D_{12}\cos(\theta_1) - D_{13}\cos(\theta_2)] + C_{23}\sin(\theta_2 - \theta_1) \tag{3.4b}$$

$$x = -D_{23}\cos(\theta_1 - \theta_2) + \{D_{12}\cos(\theta_1) + D_{13}\cos(\theta_2)\}/2 + \{C_{12}\sin(\theta_1) + C_{13}\sin(\theta_2)\}/2 \tag{3.4c}$$

$$y = \{D_{13}\cos(\theta_2) - D_{12}\cos(\theta_1)\}/2 + C_{23}\sin(\theta_1 - \theta_2) + \{-C_{12}\sin(\theta_1) + C_{13}\sin(\theta_2)\}/2. \tag{3.4d}$$

Notice that the eigenvalues of $D_\theta f$ are

$$\lambda = \alpha \pm \left[z^2 + x^2 - y^2\right]^{1/2} \tag{3.5}$$

so that a complex-conjugate pair occurs when $z^2 + x^2 < y^2$. Note that if $C_{12} = C_{13} = C_{23}$ and $\Delta P_1 = 0$, $\Delta P_2 = 0$, then $(\theta_1, \theta_2) = (0,0)$ is an equilibrium point and $D_\theta f$ has a pair of eigenvalues located at $\lambda = \alpha = D_{12} + D_{13} + D_{23}$.

In Table I, we illustrate various steady-state conditions for different system parameter values. Notice the change in eigenvalues of $D_\theta f$ as we progress from Case 1 through Case 5. The sequence of Cases 2–5 shows a clear transition from a stable to an unstable system. The equilibria of Cases 4 and 5 are unstable. Notice that Cases 1 and 4 correspond to bifurcation points, that is points of incipient flutter instability. It is interesting to note that in several computational experiments in which the system parameters $(\Delta P_1, \Delta P_2)$ were varied we could not locate unstable neighbors to Case 1. However, (3.5) indicates that with appropriate choices of parameters x, y, z they do exist. We will see later that Case 1 is generic in three (or more)-parameter families. Case 4 is much different. We easily find unstable neighbors. This case is generic in one-parameter families.

IV. Energy Functions and Hamiltonians

4.1. Energy Functions and Symmetrizing Matrices

When a dynamical system possesses only nonconservative affects of dissipative type, then it is common to associate with it the energy function of its conservative counterpart. With other types of nonconservative effects, such as the circulatory forces induced by transfer conductances, this procedure is not meaningful. How then should the energy function be defined? The essential issues are most easily described in terms of the linearized system. In this case a partial answer has been given by Kwatny *et al.* [5].

In the absence of damping, the linearized classical model is of the form

$$M\ddot{x} + Kx = 0 \tag{4.1}$$

where M, K are real and M is symmetric and positive definite. If K has real eigenvalues and a complete set of eigenvectors, then there exists a symmetric, positive definite matrix S such that $SM^{-1}K$ is a symmetric matrix. Furthermore, the equation of motion (4.1) is derivable from a Lagrangian

$$L(\dot{x}, x) = (1/2)\{\dot{x}^t S\dot{x} - x^t SM^{-1}Kx\} \tag{4.2a}$$

and has associated with it a Jacobi first integral, or energy function $E(\dot{x}, x)$, and a Hamiltonian $H(p, x)$, respectively,

$$E(\dot{x}, x) = (1/2)\{\dot{x}^t S\dot{x} + x^t SM^{-1}Kx\} \tag{4.2b}$$

$$H(p, x) = (1/2)\{p^t S^{-1}p + x^t SM^{-1}Kx\}, \quad \text{where } p = S^{-1}\dot{x}. \tag{4.2c}$$

It is important to note that the symmetrizing matrix S (and hence the Lagrangian, energy function and Hamiltonian) is not unique. In fact, let U be the transformation matrix composed of the eigenvectors of $M^{-1}K$, then the symmetrizing matrix can always be written

$$S = U\Sigma U^t, \qquad \Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n)$$

where Σ is an arbitrary nonsingular diagonal matrix. Note also that S need not be positive definite although it may always be chosen so.

In the event that K has complex eigenvalues (in which case the origin of system (4.1) is unstable) the symmetrizing matrix is no longer available as a means of constructing an energy function. However, a simple computation shows that the eigenvalues of the second-order system (4.1) occur in groups of four types: real pairs $(\sigma, -\sigma)$, purely imaginary pairs $(j\omega, -j\omega)$, quadruples $(\pm\sigma, \pm j\omega)$, and zero eigenvalues. It follows that the linear system (4.1) is equivalent to a Hamiltonian system (with quadratic Hamiltonian) under a linear transformation of coordinates. This is true for any real matrix K, regardless of the location of its eigenvalues. Thus, it makes sense to identify the energy function of (4.1) with the Hamiltonian function of an associated Hamiltonian system. We explore this point of view below.

4.2. Normal Forms of Quadratic Hamiltonians

Our purpose is to examine the structure of the energy function for general second-order systems of the type (4.1) by applying standard results for Hamiltonian systems. Consider a dynamical system characterized by a quadratic Hamiltonian in the form

$$H(y) = \frac{1}{2}y^t Qy \tag{4.3a}$$

where $y = (q_1, \cdots, q_n, p_1, \cdots, p_n)^t$ and the dynamical equations are

$$\dot{y} = JQy \tag{4.3b}$$

where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

It is well known (see Gantmacher [20, chap. 4]) that a linear transformation $y = Tz$ preserves the canonical structure of the Hamiltonian system (4.3) if and only if the transformation matrix T is a *generalized symplectic* matrix, i.e.,

$$T^t J T = cJ, \qquad c \neq 0. \tag{4.4}$$

The constant c is called the valence of the transformation. If $c = 1$, the transformation is called univalent and the matrix T is symplectic. It is reasonable then to define two Hamiltonian systems to be equivalent if they are related by a generalized symplectic transformation of coordinates. Thus any Hamiltonian system of the type (4.3) belongs to an equivalence class of Hamiltonian systems whose members are related by generalized symplectic coordinate transformations. Therefore, it is useful to seek normal form representations for these classes. Arnold [21] provides a complete classification attributed to Galin following Williamson [22] which gives just such a normal form characterization of quadratic Hamiltonians. An equivalent classification, also developed independently from Williamson's fundamental papers, is given by Laub and Meyer [23].

In the present analysis, we are interested in a very special situation. For an undamped power system undergoing load parameter variations, stability of the equilibrium is determined by the linearized equations (4.1). For a stable equilibrium, all of the eigenvalues are on the imaginary axis. Loss of stability occurs when a pair of imaginary eigenvalues meet and move off of the imaginary axis. When the meeting takes place at a point $j\omega \neq 0$ the loss of stability of the equilibrium is typically accompanied by the presence of a limit cycle.[12] This is the situation of interest. Thus, we wish to analyze a transition of the eigenvalue pattern through the sequence: $(\pm j\omega_1, \pm j\omega_2) \rightarrow (\pm j\omega)^2 \rightarrow (\pm \sigma, \pm j\omega)$. To do this we consider the normal form and universal perturbation of the Hamiltonian corresponding to the degenerate pattern $(\pm j\omega)^2$.

Williamson [22] identifies four[13] nonequivalent Hamiltonian normal forms giving rise to this eigenvalue pattern:

$$H_1 = \frac{1}{2}\left\{\left(p_1^2 + \omega^2 q_1^2\right) + \left(p_2^2 + \omega^2 q_2^2\right)\right\} \tag{4.5a}$$

$$H_2 = \frac{1}{2}\left\{\left(p_1^2 + \omega^2 q_1^2\right) - \left(p_2^2 + \omega^2 q_2^2\right)\right\} \tag{4.5b}$$

$$H_3 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \omega\left(q_1 p_2 - q_2 p_1\right) \tag{4.6a}$$

$$H_4 = -\frac{1}{2}\left(p_1^2 + p_2^2\right) + \omega\left(q_1 p_2 - q_2 p_1\right). \tag{4.6b}$$

We point out that the elementary divisors corresponding to the dynamical systems (4.5) are $(\lambda \pm j\omega)$, whereas those of (4.6) are $(\lambda \pm j\omega)^2$. Note that the Hamiltonians (4.5a) and (4.5b) correspond to stable systems, however, in all but the first case the Hamiltonian or "energy" function is not definite and cannot be used as a Lyapunov function. The Hamiltonian (4.5a) is simply the sum of two noninteracting harmonic oscillators. In (4.5b) the second oscillator is running backwards (see Abraham and Marsden [24, chap. 3]). We will return to this point later.

In general, if λ is an eigenvalue of JQ, then so is $-\lambda$, λ^*, and $-\lambda^*$, and it is possible to construct a real, invariant space, denoted by I_λ, as the union of the eigenspaces of the quadruple of eigenvalues $(\lambda, -\lambda, \lambda^*, -\lambda^*)$. Moreover, I_λ is a symplectic subspace and motion on it is governed by the restricted quadratic Hamiltonian denoted $H|I_\lambda$ (see McKay [26] for further details). A real symplectic vector space with a given quadratic form H can be decomposed into a direct sum of real symplectic subspaces, I_λ, so that the form H is represented as the sum of normal forms for $H|I_\lambda$ on these subspaces.

By a suitable choice of coordinates in I_λ, the quadratic form can be reduced to the sum of squares. The pair of numbers of positive and negative squares is called the *signature* of the quadratic form[14] and it plays an important role in the stability analysis of Hamiltonian systems. We will designate the signature of $H|I_\lambda$ by the two-tuple $\text{sig}_\lambda = (\text{no. positive squares, no. negative squares})$. The signature is said to be positive, negative, or mixed if, respectively, the number of negative squares is zero, the number of positive squares is zero, or both numbers are nonzero. The signature of H_1 is (4,0), whereas H_2, H_3, H_4 all have signature (2,2). MacKay [26] notes some interesting properties related to the signature of Hamiltonian systems. The concept of signature has important implications although it is not widely used in engineering circles. Although it is, perhaps, most common to deal with Hamiltonians of positive signature, this situation is by no means universal. The restricted three-body problem is a well-known counterexample.[15]

Example Revisited:

Consider the example of Section III and Case 4 in particular. Recall that the linearized system has eigenvalues $\pm j\omega$, $\omega = (1.5)^{1/2}$, each with algebraic multiplicity two and geometric multiplicity one. Furthermore, the linearized system can be put in the first-order form

$$\dot{y} = Ay, \qquad A = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}, \qquad y = \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix}. \tag{4.7}$$

Suppose that the generalized eigenvectors of A corresponding to $\lambda = j\omega$ are $h = a + jb$, $g = c + jd$ with a, b, c, d real vectors. Define the transformation matrix

[12] Such bifurcation points usually correspond to either subcritical or supercritical Hopf bifurcations.

[13] Actually, Williamson identifies five. The additional Hamiltonian is the same as H_2 with the variable subscripts interchanged.

[14] This definition is due to MacKay [26]. An equivalent definition is given by Moser [27]. Gantmacher [28, chap. 10], defines the signature of a quadratic form as the difference between the number of positive and the number of negative squares.

[15] Moser [32, sect. 4] comments on precisely this point.

$T=[a \quad b \quad c \quad d]$. Direct calculation verifies that

$$T^{-1}AT=\begin{bmatrix}\omega J_2 & I_2\\ 0 & \omega J_2\end{bmatrix} \tag{4.8}$$

which corresponds to the Hamiltonian (4.6b). If we use the transformation matrix $T=[a \quad -b \quad c \quad -d]$ the resulting system corresponds to (4.6a).

4.3. *Universal Perturbations of Quadratic Hamiltonians*

A generic individual Hamiltonian system does not have multiple eigenvalues. On the other hand, multiple eigenvalues do occur stably in parameterized families of Hamiltonian systems. Two questions arise: what are the generic multiplicities to be found in a k-parameter family of Hamiltonian systems? and what is the universal perturbation (unfolding) associated with a given degeneracy? Some results along these lines, again due to Galin, are noted by Arnold [10], [21].

The main purpose of this section is to show that the Hamiltonians (4.5) occur generically in three parameter families of quadratic Hamiltonians, whereas those of (4.6) occur generically in one parameter families and to determine the universal perturbations which characterize neighboring Hamiltonians. Once having done this, we will show that perturbations of H_1 remain stable. That is, the eigenvalues remain on the imaginary axis for arbitrary small perturbations. As will be shown this is not true of the others, perturbations of which can result in quartets of complex eigenvalues with nonzero real parts. Thus we expect to see bifurcations at equilibria characterized by the degenerate Hamiltonians of (4.5b) and (4.6).

Our approach is as follows. We consider a degenerate Hamiltonian, $H_0(y)=(1/2)y^tQ_0y$, such as those of (4.5) or (4.6), and add to it arbitrary "small" perturbations of the form $\delta H=(1/2)y^t\delta Qy$. Some of these Hamiltonians are equivalent to H_0 in the sense that they can be generated from it by a symplectic change of coordinates. Others are not. Alternatively, perturbations which produce equivalent Hamiltonians are removable by a symplectic change of coordinates. The others are nonremovable. We wish to characterize those (nonremovable) perturbations which produce Hamiltonians that are not equivalent to H_0. Consider infinitesimal generalized symplectic transformations of the form $T=I+\delta T$. The action of such a transformation on H_0 produces

$$H_0(Tz)=\frac{1}{2}z^t(I+\delta T)^tQ_0(I+\delta T)z$$

$$=\frac{1}{2}z^t\{Q_0+(\delta T^tQ_0+Q_0\delta T)+O(2)\}z. \tag{4.9}$$

Thus Hamiltonians equivalent to H_0 correspond to perturbations of the form $\delta Q=(\delta T^tQ_0+Q_0\delta T)+O(2)$. Because T is symplectic, the class of admissible infinitesimal transformations, δT, is restricted by (4.4). Thus

$$(I+\delta T)^tJ(I+\delta T)=J+\delta T^tJ+J\delta T+O(\delta T^2)=cJ$$

so that

$$T_1^tJ+JT_1=(c-1)J \tag{4.10}$$

where T_1 denotes the first-order approximation of δT. A simple calculation shows that (4.10) implies that T_1 is of the form

$$T_1=\begin{bmatrix}A & B\\ C & -A^t+\gamma I\end{bmatrix}$$

$$B=B^t, \qquad C=C^t, \qquad \gamma=c-1. \tag{4.11}$$

Thus if we partition Q_0 in the form

$$Q_0=\begin{bmatrix}R & S\\ S^t & V\end{bmatrix}, \qquad R=R^t, \qquad V=V^t \tag{4.12}$$

the removable perturbation can be obtained

$$Q_1=\begin{bmatrix}RA+SC & RB+S(-A^t+\gamma I)\\ S^tA+VC & S^tB+V(-A^t+\gamma I)\end{bmatrix}$$

$$+\begin{bmatrix}A^tR+CS^t & A^tS+CV\\ BR+(-A^t+\gamma I)S^t & BS+(-A+\gamma I)V\end{bmatrix}. \tag{4.13}$$

We are now in a position to prove the following theorem.

Theorem 4.1: The Hamiltonians H_1 and H_2 defined in (4.5) occur generically in three parameter families and their general perturbations (universal unfoldings) may be expressed

$$\delta H_1=\mu_1q_1p_2+\mu_2p_1p_2+\mu_3p_2^2$$

$$\delta H_2=\mu_1q_1p_2+\mu_2p_1p_2+\mu_3p_2^2.$$

The Hamiltonians H_3 and H_4 of (4.6) occur generically in one-parameter families and their general perturbations may be expressed

$$\delta H_3=\mu q_2^2$$

$$\delta H_4=\mu q_2^2.$$

Proof: Note that we can write the Hamiltonian systems of (4.5) in the form of (4.3) by identifying

$$Q_0=\begin{bmatrix}\omega^2E & 0\\ 0 & E\end{bmatrix}, \qquad \text{where } E=\begin{bmatrix}1 & 0\\ 0 & \rho\end{bmatrix} \qquad \text{and } \rho=\pm 1 \tag{4.14}$$

for (4.5a) or (4.5b), respectively. Then using (4.21), removable perturbations take the form:

$$Q_1=\begin{bmatrix}\omega^4(A^tE+EA) & (CE+\omega^4EB)\\ (CE+\omega^4EB)^t & -(AE+EA^t)+\gamma E\end{bmatrix}$$

$$=\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12}^t & Q_{22}\end{bmatrix} \tag{4.15}$$

where A is an arbitrary, real 2×2 matrix, B and C are arbitrary symmetric 2×2 matrices and γ is an arbitrary real number ($\gamma\neq 1$). It follows that Q_{11} may be specified as an arbitrary symmetric matrix, Q_{12} may be arbitrarily specified except for either one of its off diagonal elements, and one of the diagonal elements of symmetric matrix. Q_{22}

may be specified. It follows that the general nonremovable perturbation of Q_0 may be chosen as

$$\delta Q = \begin{bmatrix} 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 \\ \mu_1 & 0 & \mu_2 & 2\mu_3 \end{bmatrix}. \tag{4.16}$$

Thus H_1 and H_2 occur generically in three-parameter families of the form

$$H_1 = \frac{1}{2}\{(p_1^2 + \omega^2 q_1^2) + (p_2^2 + \omega^2 q_2^2)\} + \mu_1 q_1 p_2 + \mu_2 p_1 p_2 + \mu_3 p_2^2 \tag{4.17a}$$

$$H_2 = \frac{1}{2}\{(p_1^2 + \omega^2 q_1^2) - (p_2^2 + \omega^2 q_2^2)\} + \mu_1 q_1 p_2 + \mu_2 p_1 p_2 + \mu_3 p_2^2. \tag{4.17b}$$

In the case of the Hamiltonians (4.6a) and (4.6b), we have

$$Q_0 = \begin{bmatrix} 0 & \omega J \\ -\omega J & \rho I \end{bmatrix}, \qquad \rho = \pm 1. \tag{4.18}$$

Upon application of (4.13)

$$Q_1 = \begin{bmatrix} \omega(JC - CJ) & \omega(A'J - JA') + \rho C + \omega\gamma J \\ \omega(AJ - JA) + \rho C - \omega\gamma J & \omega(BJ - JB) - \rho(A + A') \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}. \tag{4.19}$$

Since A is an arbitrary, real 2×2 matrix, B and C are arbitrary symmetric 2×2 matrices, and γ is an arbitrary real number ($\gamma \neq 1$), it is not difficult to argue that Q_{22} may be specified as an arbitrary 2×2 symmetric matrix, Q_{12} may be specified as an arbitrary 2×2 matrix and the 2×2 symmetric matrix Q_{11} may be arbitrarily specified except for precisely one of the diagonal elements. It follows that the general nonremovable perturbation of Q_0 may be chosen as

$$\delta Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.20}$$

so that

$$H_3 = \frac{1}{2}(p_1^2 + p_2^2) + \omega(q_1 p_2 - q_2 p_1) + \mu q_2^2 \tag{4.21a}$$

$$H_4 = -\frac{1}{2}(p_1^2 + p_2^2) + \omega(q_1 p_2 - q_2 p_1) + \mu q_2^2. \tag{4.21b}$$

Remark on the Associated Lagrangian System:

Some further insight into the physical meaning of the Hamiltonians (4.5), (4.6), (4.17), and (4.21) may be obtained by computing the equivalent Lagrangian dynamical system. For example, applying the standard Legendre transformation to (4.21a) leads to the Lagrangian

$$L(\dot{q}, q) = \frac{1}{2}(\dot{q}_1 + \omega q_2)^2 + \frac{1}{2}(\dot{q}_2 - \omega q_1)^2 - \mu q_2^2 \tag{4.22}$$

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + 2\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \omega^2 \begin{bmatrix} -1 & 0 \\ 0 & -1 + 2\mu \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.23}$$

Incipient flutter instability corresponds to $\mu = 0$, and (4.23) is stable for $\mu > 0$ and unstable for $\mu < 0$. Notice that this system corresponds to a pair of unstable oscillators which are gyroscopically coupled, and thereby stabilized if $\mu > 0$. We may also easily compute the Jacobi first integral

$$E(\dot{q}, q) = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \omega^2 q_1^2 - (\omega^2 - 2\mu) q_2^2. \tag{4.24}$$

Thus we readily see from either (4.21a) or (4.24) that energy is not a sign definite function.

V. First Integrals and Stability

5.1. Stability Under Parameter Variations

The key observation is the following.

Theorem 5.1: The origin of the Hamiltonian system (4.3) can lose stability under parameter variations by collision of a pair of eigenvalues at a point $j\omega \neq 0$ if and only if $H|I_{j\omega}$ has mixed signature.

Proof: Stability properties under parameter variations may be analyzed directly by considering the characteristic polynomials associated with the dynamical systems defined by the Hamiltonians of (4.18) and (4.23). The characteristic polynomial associated with (4.18) is

$$\alpha(s) = s^4 + 2\omega^2(1 + \rho\mu_3)s^2 + (\omega^4 + \rho\omega^2(-\mu_1^2 - \omega^2\mu_2^2 + 2\omega^2\mu_3)) \tag{5.1}$$

where $\rho = \pm 1$. From (5.1) we obtain the eigenvalues

$$s^2 = -\omega^2(1 + \rho\mu_3) \pm \sqrt{\omega^4\mu_3^2 + \rho\omega^2(\mu_1^2 + \omega^2\mu_2^2)}. \tag{5.2}$$

It is clear from (5.2) that the nominal system, i.e., $\mu_1 = \mu_2 = \mu_3 = 0$, has a pair of eigenvalues at $\pm j\omega$, and, furthermore, it can be destabilized by arbitrarily small perturbations only for the case $\rho = -1$.

Similar calculations show that either of the nominal systems of (4.23) can be destabilized by arbitrarily small variation of the single parameter, μ. Thus destabilization can occur for arbitrarily small parameter variations only for the cases of mixed signature. Various forms of this result have been implicit in the work of many investigators (McKay [26], Moser [27]). Analogous results for periodic systems are more widely known (see, e.g. Hale [29, chap. 3]). The above result clearly illustrates the nonequivalence

of the behavior of the Hamiltonian systems of (4.5) and (4.6) under Hamiltonian perturbations.

5.2. Quadratic First Integrals

Stability analysis of Hamiltonian systems is often conducted directly, based on Lyapunov theorems, by making use of the fact that the Hamiltonian function is a first integral of the equations (4.3). Direct computation verifies the well-known fact that

$$\frac{d}{dt}H = 0. \tag{5.3}$$

It follows that a sufficient condition for stability is that H is definite. The systems defined in (4.5), both of which are stable (in fact they have the same equations of motion), make it obvious that definiteness of H is not a necessary condition for stability. If the Hamiltonian, or energy function, is not definite it is still possible to identify "natural" Lyapunov functions by combining with H other first integrals associated with (4.3). This approach traces back to Chetaev [30] and has been subsequently considered by many investigators including Rubanovskii and Stepanov [31]. We cite the latter because they present a clear example of the application of this method to the study of stability under variation of a number of parameters.

Any real scalar function $\phi(y)$ which satisfies the condition $\dot{\phi}(y) = 0$ along trajectories of (4.3) is a first integral of (4.3). We will develop a parameterization of the quadratic first integrals associated with (4.3). A quadratic first integral is of the form $\phi = y^t \Phi y$, where Φ is a real symmetric matrix. A direct calculation shows that $\dot{\phi} = 0$ if and only if

$$(JQ)^t\Phi + \Phi(JQ) = 0. \tag{5.4}$$

We seek to characterize those matrices Φ which satisfy (5.4). Gantmacher [28] provides a complete analysis of the equation

$$A\Phi - \Phi B = 0. \tag{5.5}$$

The special case of interest here, $A = (JQ)^t$ and $B = -A^t$,

$$A\Phi + \Phi A^t = 0 \tag{5.6}$$

can be carried somewhat farther because of the Hamiltonian character of A. We follow the analysis of [28] to prove the following theorem.

Theorem 5.2: A quadratic function $\phi(y) = y^t\Phi y$ is a first integral of the Hamiltonian system $H(y) = y^tQy$ if and only if the real symmetric matrix Φ is of the form

$$\Phi = USU^t$$

where U is the transformation matrix (to Jordan form) of generalized eigenvectors of $(JQ)^t$, and S is defined as follows. S is partitioned into blocks S_{ij}, $i, j = 1, \cdots, p$ compatible with the Jordan form of $(JQ)^t$. The only nonzero blocks of S_{ij} correspond to values of i and j for which $\lambda_i = \lambda_j$. These nontrivial blocks, S_{ij}, of dimension $d_i \times d_j$, are of the form

$$S_{ij} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdot & a_{d_i} \\ -a_2 & -a_3 & \cdot & -a_{d_i} & 0 \\ a_3 & \cdot & a_{d_i} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \pm a_{d_i} & 0 & \cdot & \cdot & 0 \end{bmatrix} = S_{d_i}, \quad \text{if } d_i = d_j$$

$$\text{If } d_i < d_j,\ S_{ij} = \begin{bmatrix} S_{d_i} & 0 \end{bmatrix} \text{ and if } d_i > d_j,\ S_{ij} = \begin{bmatrix} S_{d_i} \\ 0 \end{bmatrix}.$$

Proof: Note that if Φ is any matrix which satisfies (5.6), not necessarily real or symmetric, then its real symmetric part also satisfies (5.6). Thus we need only show that all solutions satisfying (5.6) are of the form stated in the theorem. Let U denote the transformation matrix of generalized eigenvectors of A so that

$$A = UA_JU^{-1}$$

where A_J is the Jordan form of A. Suppose that A has precisely p Jordan blocks of dimensions d_i, $d_1 + \cdots + d_p = 2n$. It follows that

$$A_J = \operatorname{diag}\left(J_{\lambda_1}, \cdots, J_{\lambda_p}\right)$$

and

$$A^t = VA_JV^{-1}, \qquad V^{-1} = PU^t.$$

P is the permutation matrix:

$$P = \operatorname{diag}\left(P_1, \cdots, P_p\right)$$

where P_i is a $d_i \times d_i$ matrix of the form

$$P_i = \begin{bmatrix} 0 & \cdot\cdot 0 & 1 \\ 0 & \cdot\cdot 1 & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 1 & 0\cdot\cdot & 0 \end{bmatrix}.$$

Now, (5.6) reduces to

$$A_J\tilde{\Phi} + \tilde{\Phi}A_J = 0 \tag{5.7}$$

where

$$\Phi = U\tilde{\Phi}V^{-1} = U\tilde{\Phi}PU^t. \tag{5.8}$$

$\tilde{\Phi}$ can then be partitioned compatibly with A_J into p^2 blocks $\tilde{\Phi}_{ij}$ $(i, j = 1, \cdots, p)$, and (5.7) written as p^2 equations

$$J_{\lambda_i}\tilde{\Phi}_{ij} + \tilde{\Phi}_{ij}J_{\lambda_j} = 0, \qquad i, j = 1, \cdots, p. \tag{5.9}$$

Repeating the calculations in [28] it can be shown that if $\lambda_i \neq -\lambda_j$ then $\tilde{\Phi}_{ij} = 0$. If $\lambda_i = -\lambda_j$, then $\tilde{\Phi}_{ij}$ is of a special triangular form. If the block is square, i.e., $d_i = d_j$, then

$$\tilde{\Phi}_{ij} = \begin{bmatrix} a_{d_i} & \cdot & a_3 & a_2 & a_1 \\ 0 & -a_{d_i} & \cdot & -a_3 & -a_2 \\ \cdot & \cdot & a_{d_i} & \cdot & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \pm a_{d_i} \end{bmatrix} = T_{d_i}. \tag{5.10a}$$

Observe the alternating signs along the diagonals. If $d_i < d_j$

$$\tilde{\Phi}_{ij} = \begin{bmatrix} 0 & T_{d_i} \end{bmatrix}. \tag{5.10b}$$

If $d_i > d_j$,

$$\tilde{\Phi}_{ij} = \begin{bmatrix} T_{d_i} \\ 0 \end{bmatrix}. \tag{5.10c}$$

Note also that if $d_i = d_j$, then

$$[\Phi P]_{ij} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdot & a_{d_i} \\ -a_2 & -a_3 & \cdot & -a_{d_i} & 0 \\ a_3 & \cdot & a_{d_i} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \pm a_{d_i} & 0 & \cdot & \cdot & 0 \end{bmatrix} = S_{d_i}. \tag{5.11a}$$

If $d_i < d_j$,

$$[\tilde{\Phi} P]_{ij} = \begin{bmatrix} S_{d_i} & 0 \end{bmatrix}. \tag{5.11b}$$

If $d_i > d_j$,

$$[\tilde{\Phi} P]_{ij} = \begin{bmatrix} S_{d_i} \\ 0 \end{bmatrix}. \tag{5.11c}$$

Corollary 5.1: If JQ has no eigenvalues at the origin, then the number of independent real symmetric matrices[16] of the form specified in Theorem 5.2 is

$$N = \sum_{i \in I} m_i q_i + \sum_{i \in I_o} m_i q_i$$

where m_i and q_i are the algebraic and geometric multiplicities, respectively, of the r distinct eigenvalues of JQ, I denotes the index set of the (distinct) eigenvalues on the positive imaginary axis, and I_o denotes the index set of the (distinct) eigenvalues in the open right half plane.

Proof: Since A is real, its complex eigenvalues and corresponding generalized eigenvectors occur in conjugate pairs. Furthermore, A is Hamiltonian so that eigenvalues occur, with compatible multiplicities and Jordan blocks, as: real pairs $\lambda, -\lambda$, imaginary pairs $\lambda, \lambda^* = -\lambda$, and strictly complex quartets $\lambda, -\lambda, \lambda^*, -\lambda^*$. The real symmetric solutions Φ of (5.6) are obtained if and only if the following conditions hold: 1) if $\lambda_i = -\lambda_j$, then $S_{ij} = S_{ji}^t$ (which implies $S_{d_i} = S_{d_j}^t$), 2) if $\lambda_j = \lambda_i^*$, then $S_{ij} = S_{ji}^*$, 3) if $\lambda_j = -\lambda_i^*$, then $S_{ij} = (S_{ji}^*)^t$.

Suppose A has precisely r distinct eigenvalues λ_i, $i = 1, \cdots, r$. Let m_i denote the algebraic multiplicity of λ_i and q_i its geometric multiplicity. There exists an eigenvalue $-\lambda_i$ with precisely the same multiplicities. Thus, any eigenvalue λ and its associate $-\lambda$ each contribute q Jordan blocks to A_J of dimension d_k. For convenience consider $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$. Accordingly, this pair of eigenvalues produces q^2 nontrivial blocks S_{ij}, of dimension $d_i \times d_j$, $i = 1, \cdots, q$ and $j = q+1, \cdots, 2q$, and a companion set with $i = q+1, \cdots, 2q$ and $j = 1, \cdots, q$. Note that $d_1 + \cdots + d_q = m$. Each block contains $\min(d_i, d_j)$ parameters. Moreover, these parameters are arbitrary except that $S_{ij} = S_{ji}^t$. (or $(S_{ji}^*)^t$ if λ is pure imaginary). It follows that the number of independent constants (complex in the event λ is complex) associated with the pair of eigenvalues $\lambda, -\lambda$ is

$$N_\lambda = \sum_{i=1}^{q} \sum_{j=1}^{q} \min(d_i, d_j).$$

Assume, without any loss in generality, that the blocks are ordered via size, so that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_q$. Then

$$N_\lambda = \sum_{i=1}^{q} \left(\sum_{j=1}^{i} d_j + \sum_{j=i+1}^{q} d_i \right)$$

$$= \sum_{i=1}^{q} \left(\sum_{j=1}^{q} d_i \right) = \left(\sum_{i=1}^{q} d_i \right) q = mq.$$

If λ is strictly complex, then there is a corresponding and distinct pair of eigenvalues λ^* and $-\lambda^*$, but these contribute constants which are simply the complex conjugates of those of λ. Each such complex constant corresponds to two independent real constants and hence if λ is complex we may associate N_λ real constants with λ and an equal number of real constants with λ^*. It follows that the number of independent, real, symmetric solutions of (5.6) is

$$N = \sum_{i \in I} m_i q_i + \sum_{i \in I_o} m_i q_i. \tag{5.12}$$

We now consider two examples of the application of these results, corresponding to the Hamiltonians (4.5) and (4.6a).

Consider the Hamiltonian system (4.5). We will show that it is possible to construct a positive definite quadratic first integral, and hence a Lyapunov function for either Hamiltonian. In this case we have

$$JQ = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \rho \\ -\omega^2 & 0 & 0 & 0 \\ 0 & -\omega^2 \rho & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\pm j\omega$, each with algebraic and geometric multiplicity 2. The matrix of eigenvectors of $(JQ)^t$ is

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \dfrac{-j}{\omega} & 0 & \dfrac{j}{\omega} & 0 \\ 0 & \dfrac{-j\rho}{\omega} & 0 & \dfrac{j\rho}{\omega} \end{bmatrix}.$$

According to Theorem 5.2 we compose the matrix

$$S = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ a^* & c^* & 0 & 0 \\ b^* & d^* & 0 & 0 \end{bmatrix}$$

[16] Note that the number of independent matrices characterized herein is not the same as the number of independent quadratic first integrals. The latter is typically smaller and never greater. This is because the vector fields generated by the gradients of two quadratic first integrals may not be independent even though the matrices defining the quadratic forms are themselves independent.

where we have been mindful of the fact that real, symmetric Φ obtains only when $S_{ij} = S_{ji}^t$ if $\lambda_i = -\lambda_j$ and $S_{ij} = S_{ji}^*$ if $\lambda_j = \lambda_i^*$. Thus we obtain

$$\Phi = USU^t = \begin{bmatrix} \omega^2 a & \omega^2 b & 0 & -\omega\beta\rho \\ \omega^2 b & \omega^2 d & \omega\beta & 0 \\ 0 & \omega\beta & a & b\rho \\ -\omega\beta\rho & 0 & b\rho & d \end{bmatrix},$$

$$a, b, d, \beta = \text{arbitrary real constants.} \quad (5.13)$$

Upon comparison with Q_0 in (4.14) it is readily checked that the Hamiltonian is a first integral. Note that Φ in (5.13) is easily made positive definite, for example choose $a, d > 0$ and $b, \beta = 0$. Thus we construct a Lyapunov function for either Hamiltonian system (4.5a) or (4.5b) from its quadratic first integrals. This result may be somewhat surprising for (4.5b) since, although this system is stable, it is structurally unstable even when viewed within the class of Hamiltonian systems.

We will construct the general quadratic first integral for the Hamiltonian (4.6a) and show that, as expected, a definite one does not exist. Then, we will examine the general perturbation of (4.6a) as given by (4.21). It will be shown that a definite quadratic first integral does exist (even though the Hamiltonian itself is not definite) when the perturbed system is stable. For the case of Hamiltonian (4.6a), we have

$$JQ = \begin{bmatrix} -\omega J & I \\ 0 & -\omega J \end{bmatrix}.$$

The eigenvalues are $\pm j\omega$ each having algebraic multiplicity two and geometric multiplicity one. Thus in accordance with Corollary 5.1, we anticipate two independent matrices Φ. The matrix of generalized eigenvectors of $(JQ)^t$ is also easily computed to be

$$U = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & j & 0 & -j \\ 1 & 0 & 1 & 0 \\ j & 0 & -j & 0 \end{bmatrix}.$$

As specified in Theorem 5.2 we compose the matrix

$$S = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & 0 \\ a^* & -b^* & 0 & 0 \\ b^* & 0 & 0 & 0 \end{bmatrix}$$

where we require $S_{ij} = S_{ji}^t$ if $\lambda_i = -\lambda_j$ and $S_{ij} = S_{ji}^*$ if $\lambda_j = \lambda_i^*$. Thus we obtain

$$\Phi = USU^t = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \\ 0 & -\beta & \alpha & 0 \\ \beta & 0 & 0 & \alpha \end{bmatrix},$$

$$\alpha, \beta = \text{arbitrary real constants.} \quad (5.14)$$

Compare this result with Q_0 in (4.18). Observe that it is not possible to choose α, β so that Φ is definite.

Let us now consider the perturbed Hamiltonian (4.22):

$$Q = \begin{bmatrix} 0 & 0 & 0 & \omega \\ 0 & 2\mu & -\omega & 0 \\ 0 & -\omega & \rho & 0 \\ \omega & 0 & 0 & \rho \end{bmatrix}$$

which reduces to (4.6) when $\mu = 0$, for which the general quadratic first integral is given by (5.14). Consider the situation in which $\mu \neq 0$. In this case, $(JQ)^t$ has eigenvalues

$$\lambda^2 = -(\omega^2 + \mu\rho) \pm [\mu\rho(4\omega^2 + \mu\rho)]^{1/2}.$$

Notice that the eigenvalues are distinct and purely imaginary if $\mu\rho > 0$ and sufficiently small. In this case the system is stable. On the other hand, $\mu\rho < 0$ results in an unstable system. We consider the former situation and set

$$\mu\rho = \epsilon^2\omega^2$$

where ϵ is a small parameter. It follows that the eigenvalues are now given by

$$\lambda = \pm jf(\epsilon)\omega, \qquad f(\epsilon) = 1 \pm \epsilon + \frac{1}{2}\epsilon^2 + O(\epsilon^3).$$

The matrix of eigenvectors is directly computed:

$$U = \begin{bmatrix} -\frac{j\epsilon(1-\epsilon)\omega}{\epsilon\omega} & \frac{j\epsilon(1-\epsilon)\omega}{\epsilon\omega} & -\frac{j\epsilon(1+\epsilon)\omega}{\epsilon\omega} & \frac{j\epsilon(1+\epsilon)\omega}{\epsilon\omega} \\ 1-\frac{\epsilon}{2} & 1-\frac{\epsilon}{2} & -1-\frac{\epsilon}{2} & -1-\frac{\epsilon}{2} \\ j\left(1-\frac{\epsilon}{2}\right) & -j\left(1-\frac{\epsilon}{2}\right) & -j\left(1+\frac{\epsilon}{2}\right) & j\left(1+\frac{\epsilon}{2}\right) \end{bmatrix} + \text{h.o.t.}$$

We also obtain the general form of S

$$S = \begin{bmatrix} 0 & a & 0 & 0 \\ a^* & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^* & 0 \end{bmatrix}$$

and finally,

$$\Phi = USU' = A\begin{bmatrix} \epsilon^2\omega^2(1-\epsilon)^2 & 0 & 0 & -\epsilon\omega(1-\epsilon)\left(1-\frac{\epsilon}{2}\right) \\ 0 & \epsilon^2\omega^2 & \epsilon\omega\left(1-\frac{\epsilon}{2}\right) & 0 \\ 0 & \epsilon\omega\left(1-\frac{\epsilon}{2}\right) & \left(1-\frac{\epsilon}{2}\right)^2 & 0 \\ -\epsilon\omega(1-\epsilon)\left(1-\frac{\epsilon}{2}\right) & 0 & 0 & \left(1-\frac{\epsilon}{2}\right)^2 \end{bmatrix}$$

$$+ B\begin{bmatrix} \epsilon^2\omega^2(1+\epsilon)^2 & 0 & 0 & \epsilon\omega(1+\epsilon)\left(1+\frac{\epsilon}{2}\right) \\ 0 & \epsilon^2\omega^2 & -\epsilon\omega\left(1+\frac{\epsilon}{2}\right) & 0 \\ 0 & -\epsilon\omega\left(1+\frac{\epsilon}{2}\right) & \left(1+\frac{\epsilon}{2}\right)^2 & 0 \\ \epsilon\omega(1+\epsilon)\left(1+\frac{\epsilon}{2}\right) & 0 & 0 & \left(1+\frac{\epsilon}{2}\right)^2 \end{bmatrix} + \text{h.o.t.} \quad (5.15)$$

where $A = \mathrm{Re}(a)$, $B = \mathrm{Re}(b)$ are arbitrary real constants. Note that we recover the Hamiltonian (up to leading terms in ϵ) by choosing

$$A = \frac{(1+\epsilon)^2}{2\epsilon}, \qquad B = -\frac{(1-\epsilon)^2}{2\epsilon}.$$

Furthermore, we can obtain a positive definite first integral by choosing

$$A = \frac{1}{2}\left[1-\frac{\epsilon}{2}\right]^{-1}, \qquad B = \frac{1}{2}\left[1+\frac{\epsilon}{2}\right]^{-1}.$$

This leads to

$$\Phi = \begin{bmatrix} \epsilon^2\omega^2 & 0 & 0 & -\epsilon^2\omega \\ 0 & \epsilon^2\omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon^2\omega & 0 & 0 & 1 \end{bmatrix} + O(\epsilon^4). \quad (5.16)$$

The eigenvalues are seen to be $\epsilon^2\omega^2(1-\epsilon^2), \epsilon^2\omega^2, 1, 1$ plus higher order terms in ϵ.

5.3. *Remarks on the Construction of Lyapunov Functions*

Given a power system with a regular, stable equilibrium point it is of interest to characterize the extent of the domain of attraction. Various methods have been developed based on the use of a Lyapunov function. As previously noted, the specification of a candidate Lyapunov function often begins with some notion of an energy function. The appropriate structure for the energy function and hence the Lyapunov function is unclear for a power system with loads. One possibility is to begin with a local characterization of energy as a first approximation to the Lyapunov function and then to refine it in an attempt to obtain improved estimates of the relevant portions of the stability boundary. Our results show that even the characterization of local Lyapunov functions can be subtle.

Consider the linearized model in the form of (4.1), repeated here

$$M\ddot{x} + Kx = 0 \quad (5.17)$$

and assume that eigenvalues of (5.17) are $j\omega_1, \cdots, j\omega_n$ and are distinct. Then the equilibrium point is regular and stable. Moreover, as described in Section IV, it is very easy to construct an energy function using the symmetrizing matrix, S. It was noted that the specification of S is not unique and the significance of this fact is now apparent. First, S may always be chosen to be positive definite and the remaining nonuniqueness is a scaling issue. However, it is also possible to associate the stable system with a sign indefinite S which has the same meaning as the association of (5.17) with a different and nonequivalent Hamiltonian. Now, the energy function is not directly usable as a Lyapunov function and it becomes necessary to construct a suitable sign definite first integral for this purpose. Why bother? If the Lyapunov function is to characterize the true stability boundary with any fidelity we would anticipate that it should reflect the essential qualitative properties of the system. Certainly, if the system operates close to a stability boundary in the parameter space we should expect to capture that important property in the proposed Lyapunov function.

The immediate clue that the system may be operating close to a stability boundary is the proximity of the characteristic frequencies. The theory developed herein addresses the simplest case where a single pair of frequencies are close enough to warrant concern. Higher order degeneracies may be studied by similar methods. Our suggestion is that a degenerate system on the nearby stability boundary should be located by adjusting system parameters. The original system may then be imbedded in a parameterized family associated with the degenerate system and this will

naturally yield quadratic first integrals for use as candidate Lyapunov functions. Thus we begin the construction with a parameterized quadratic first integral which exhibits the required behavior under parameter variations. Recall that the degenerate Hamiltonians (4.6) are generic in one-parameter families (i.e., 4.21) and these are sufficient for situations such as Case 4 in the example of Section III.

VI. Conclusions

The results of Kwatny *et al.* [5] simply that once an energy function has been obtained for a strongly stable power system, smooth perturbations of this energy function do characterize all neighboring systems. Such a conclusion is incorrect for those exceptional systems which may be stable but are not strongly stable. These exceptional systems are significant because they correspond to points on the boundary of the domain of stable systems in the parameter space.

In this paper we study undamped power systems at points of incipient flutter instability, i.e., at equilibria characterized by a conjugate pair of purely imaginary eigenvalues of algebraic multiplicity 2 (the simplest case). We show that such systems may be associated with four nonequivalent energy functions (Hamiltonians). Moreover, we derive the universal perturbations of these Hamiltonians and show that they are generic in one or three parameter families of Hamiltonians. It is also shown that systems associated with these Hamiltonians behave qualitatively differently from each other under perturbations. In particular, it is possible for systems characterized by three of these Hamiltonians (those of mixed signature) to lose stability under perturbations, while the single Hamiltonian of positive signature is not associated with a system which will lose stability under perturbations.

These results allow us to provide a clear picture of the mechanics of the instability mechanism, that is the transition from a stable to an unstable system. We see that the conventional paradigm of a classical power system as an interconnection of simple oscillators is invalid near points of flutter instability. The system behaves more like an interconnection of gyroscopically coupled unstable oscillators. One implication of this is that stability behavior cannot be explained by the potential energy function alone.

We also note that the Hamiltonians of mixed signature associated with perturbations of these systems cannot be used as Lyapunov functions (because they are indefinite) even if the system is stable. In this case we demonstrate that it is possible to construct the general quadratic first integral and to find one which does serve as a "natural" Lyapunov function. Our results provide a method of construction which yields all quadratic first integrals associated with a linear Hamiltonian system.

The significance of a natural Lyapunov function is that it does facilitate the characterization of power system stability domains both in the state space and the parameter space. Modern power system operational issues (e.g., voltage collapse) are linked to parametric loss of stability. Our analysis, although local, serves to clarify questions of existence or nonexistence of energy functions for power systems with loads. We show that local energy functions do exist, in a sense consistent with the inverse problem of analytical mechanics, even near points of flutter instability. Unfortunately, the energy function itself may not be a Lyapunov function so that it is necessary to combine other first integrals with energy in order to construct one.

These results are significant because power systems are increasingly required to operate near stability limits. Thus it is necessary to develop tools tailored to the analysis of system stability near these limits. Our results explain how conventional energy methods must be modified in order to capture the essential features of flutter instability. We might note that the somewhat less subtle divergence instability can easily be incorporated within the framework described herein. The key element in this approach is the embedding of the linearized system into an appropriate, parameterized family which exhibits the instability mechanism. In this way we obtain candidate (quadratic) Lyapunov functions which consistently portray the instability mechanism. Such local Lyapunov functions may be extended in order to obtain improved estimates of the domain of attraction.

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